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Decompositions of a Krein space in regular subspaces invariant under a uniformly bounded C_0-semigroup of bi-contractions

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Abstract

We give necessary and sufficient conditions under which a C_0-semigroup of bi-contractions on a Krein space is similar to a semigroup of contractions on a Hilbert space. Under these and additional conditions we obtain direct sum decompositions of the Krein space into invariant regular subspaces and we describe the behavior of the semigroup on each of these summands. In the last section we give sufficient conditions for the co-generator of the semigroup to be power bounded.

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1. Introduction

Let $\{U(t)\}_{t>0}$ be a C_0-semigroup of bi-contractions on a Krein space $\mathscr{K}$. In Section 3, we pose the following question: When is this semigroup similar to a semigroup of contractions in a Hilbert space? More specifically: When does there

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exist a C_0-semigroup $\{V(t)\}_{t>0}$ of contractions on a Hilbert space $\mathscr{H}$ and a bounded and boundedly invertible operator $S : \mathscr{K} \to \mathscr{H}$ (a similarity operator for short) such that $U(t) = S^{-1}V(t)S$ for all $t>0$? Similarity is of interest because it preserves spectral properties. Theorem 3.4 and the remark following it provide various answers. They can be seen as generalizations of results in [ABDJ] where the same question is studied for a single bi-contraction. One necessary and sufficient condition is analytic: the semigroup must be uniformly bounded, that is,

$$\sup_{t>0} ||U(t)|| < \infty;$$

another is more geometric and in terms of a direct sum decomposition of $\mathscr{K}$ into an invariant maximal uniformly positive and an invariant maximal uniformly negative subspace. Here invariant means invariant under $U(t)$ for every $t>0$. Assuming these necessary and sufficient conditions we consider in Sections 4 and 5 additional conditions which give rise to other decompositions of $\mathscr{K}$ into invariant regular subspaces.

In Section 4, we assume additionally that for some $t_0>0$, $U(t_0)$ belongs to the class **H**; this class is defined at the beginning of the section. The assumption presented itself quite naturally, when it turned out (see Proposition 4.2) that there actually exists a decomposable bi-contraction X from the class **H** which commutes with each $U(t)$. The assumption then is that X coincides with $U(t_0)$. The main result in Section 4 is Theorem 4.4: $\mathscr{K}$ contains an invariant subspace $\mathscr{P}$ which, if not equal to $\{0\}$, is the orthogonal sum of finitely many invariant Pontryagin or anti-Pontryagin subspaces on which each $U(t)$ is unitary. The orthogonal complement $\mathscr{K} \ominus \mathscr{P}$ is also invariant and Theorem 3.4 can be applied. In this case the direct sum decomposition of $\mathscr{K} \ominus \mathscr{P}$ into an invariant maximal uniformly positive and an invariant maximal uniformly negative subspace is unique.

In Section 5, we assume additionally that the set $\sigma(A) \cap i\mathbb{R}$ is countable, where A is the generator of the semigroup. The main result here is Theorem 5.1: One of the summands in the decomposition of $\mathscr{K}$ into invariant subspaces is a Hilbert space on which the semigroup is stable, that is, strongly converges to 0 when $t \to \infty$. In Theorem 5.3, we assume that the semigroup has a strong limit when $t \to \infty$; one of the consequences is that $\mathscr{K}$ contains an invariant maximal uniformly negative subspace on which each $U(t)$ acts as the identity.

Section 6 stands out from the other sections. Here we do not prove statements about invariant subspaces for uniformly bounded C_0-semigroups. The uniform boundedness of a C_0-semigroup $\{U(t)\}_{t>0}$ is equivalent to one of the $U(t)$'s (and then all of them) being power bounded; see Lemma 3.1. The question we are concerned with in Section 6 is: Is the power boundedness of each member of the C_0-semigroup carried over to its co-generator? Necessary and sufficient conditions for an operator A to be the generator of a uniformly bounded C_0-semigroup are well known and recalled at the beginning of Section 3. Here we deal with the co-generator. It is sufficient to study the question in a Hilbert space environment. The answer to the question is: yes, if the generator is bounded, and yes, if the generator is invertible and the inverse is also the generator of a uniformly bounded C_0-semigroup. See Theorems 6.1 and 6.2. The latter theorem was obtained independently by Gomilko (private communication). The question if uniform boundedness of a C_0-semigroup implies power boundness of its co-generator without any assumptions on its generator is still open.

In Section 2, we consider two commuting bi-contractions T_1 and T_2 on a Krein space $\mathscr{K}$ and prove that if they are power bounded then $\mathscr{K}$ contains a maximal uniformly positive subspace and a maximal uniformly negative subspace which are invariant under T_1 and T_2. See Theorem 2.2, which is the main result of Section 2. We also prove a generalization of a result of Ando [A1] to a Krein space setting. This is Proposition 2.3 and it states that T_1 and T_2 have unitary dilations which also commute. In the proofs of Theorem 2.2 and Proposition 2.3 we use two lemmas, Lemmas 2.1 and 2.4 about the existence of commuting isometric dilations of T_1 and T_2 and the existence of commuting unitary extensions of these dilations, respectively. They are indefinite versions of the corresponding Hilbert space results, see [SNF, Theorem 1.6.1 and Proposition 1.6.2] and the original sources mentioned there. In our proof of Proposition 2.3 we follow the proof of [SNF, Theorem 1.6.4].

Theorem 2.2 provides the basis for the proof of Theorem 3.4, which as mentioned above is the main theorem of Section 3. The basic results about semigroups, generators and co-generators briefly recalled at the beginning of this section. We frequently refer to the standard book [HPh]. In this section also we prove some Krein space results such as Propositions 3.2 and 3.3, which are well-known for Hilbert spaces. The class of operators with property **H** considered in Section 4 is extensively studied in [AI] and for the results we use but do not prove, we refer to this book. The main result of Section 5, Theorem 5.1, is actually a corollary of [AB, Corollary 2.6]. In [AB] semigroups are studied in normed spaces; we specialize to a Krein space. Finally, we mention that the proofs of the two theorems in Section 6 are based on integral estimates on the norms of powers of the resolvent of the generator and the norms of powers of the co-generator of a C_0-semigroup proved by Gomilko [G1,G2].

We mention some recent papers dealing with C_0-semigroups of operators on a space with an indefinite metric: Kuzhel [K2,K3] considers such semigroups in connection with Lax–Phillips scattering in Pontryagin spaces. Vesenti [V2] proves the existence of a strongly continuous semigroup of contractions on a Pontryagin space whose generator is a given maximal dissipative operator. Chen [C] shows that a C_0-semigroup of contractions on a Pontryagin space admits a dilation to a C_0-semigroup of unitary operators and investigates the connection between the generators of these semigroups. J-unitary dilations also appear in [K1]. Finally, in [KRS,V1] semigroups on Krein spaces are used in connection with fractional linear transformations.

We assume that the reader is familiar with the geometry of indefinite inner product spaces and the corresponding operator theory; see the books [AI,Bo,IKL] and also [A2,DR]. We briefly recall some of the notions, also in order to make clear the notations used in this paper. A Krein space $\{\mathscr{K}, [\cdot,\cdot]\}$ is a complex linear space with an inner product $[\cdot,\cdot]$ such that $\mathscr{K}$ admits a fundamental decomposition $\mathscr{K} =$

$\mathcal{K}_+ \oplus \mathcal{K}_-$ in which $\{\mathcal{K}_\pm, \pm[\cdot,\cdot]\}$ are Hilbert spaces and $\mathcal{K}_+ \perp \mathcal{K}_-$. The positive/negative index $:= \dim \mathcal{K}_\pm$ is independent of such decompositions. In this note a Pontryagin space is a Krein space with a finite negative index, an anti-Pontryagin space is a Krein space with a finite positive index.

Square brackets will denote the (generally) indefinite inner product on a Krein space, which if not specified otherwise, is denoted by $\mathcal{K}$. If $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-$ is a fundamental decomposition of $\mathcal{K}$, the operator $J = P_+ - P_-$, where $P_\pm$ is the orthogonal projection onto $\mathcal{K}_\pm$, is called the associated fundamental symmetry. The Hilbert space J-inner products $(\cdot,\cdot) := [J\cdot,\cdot]$ which depend on J, give rise to equivalent norms $||\cdot||$, and by definition $\mathcal{K}$ is endowed with this norm topology. Whenever we use an inner product with brackets or a norm on $\mathcal{K}$ we tacitly assume that this is one of these inner products and norms associated with a prefixed fundamental symmetry, denoted by J, on $\mathcal{K}$.

By definition a contraction T on a Krein space $\mathcal{K}$ is a bounded operator T on $\mathcal{K}$ with the property that $[Tx, Tx] \leqslant [x, x]$ for all $x \in \mathcal{K}$. A bi-contraction T on $\mathcal{K}$ is a contraction whose adjoint T^* is a contraction also. In this paper a dissipative operator A on $\mathcal{K}$ is a closed densely defined operator A for which $\mathrm{Re}[Ax, x] \leqslant 0$, $x \in \mathrm{dom}\, A$. In [ABDJ] we used a different definition (namely $\mathrm{Im}\,[Ax, x] \geqslant 0$). In the paper we use dilations of operators as considered in for example [GGK,N]; for a discussion of dilations in a Krein space setting we refer to [DR].

In the sequel $\mathbb{R}$ and $\mathbb{C}$ are the sets of real and complex numbers, $\mathbb{D}$ is the open unit disk, and $\mathbb{T}$ is the unit circle. The closure of a set S is denoted by $\bar{S}$. The sum, the direct sum and the orthogonal sum are indicated by $+$, $\dot{+}$ and $\oplus$, respectively; $\ominus$ means the orthogonal difference. $\sigma(T)$ and $\sigma_p(T)$ stand for the spectrum and the point spectrum of an operator T. The adjoint of an operator A on a Krein or Hilbert space and the adjoint of a matrix A will be denoted by A^*; the complex conjugate of a complex number λ will also be denoted by λ^*.

2. Invariant subspaces for two commuting bi-contractions

In this section we prove that two commuting power bounded bi-contractions on a Krein space have an invariant maximal uniformly positive and an invariant maximal uniformly negative subspace in common. See Theorem 2.2, which is the main result of this section. In the next section we apply this theorem to obtain some similarity results for uniformly bounded C_0-semigroups of bi-contractions. We also prove a Krein space version of Ando's theorem in Ando [A1] (or [SNF, Theorem 1.6.4]), see Proposition 2.3 below, which is of independent interest. In the proofs of Theorem 2.2 and Proposition 2.3 we use two lemmas which are Krein space analogs of [SNF, Theorem 1.6.1 and Proposition 1.6.2]. Their proofs are similar to their Hilbert space counterparts.

Lemma 2.1. *For each $i = 1, 2$, let T_i be a contraction on a Krein space $\mathcal{K}$ and assume $T_1T_2 = T_2T_1$. Then there are a Hilbert space $\mathcal{F}$ and for each i an isometric dilation V_i*

of T_i on the Krein space $\widetilde{\mathscr{K}} := \mathscr{K} \oplus \mathscr{F}$:

$$V_i = \begin{pmatrix} T_i & 0 \\ * & * \end{pmatrix} : \begin{pmatrix} \mathscr{K} \\ \mathscr{F} \end{pmatrix} \to \begin{pmatrix} \mathscr{K} \\ \mathscr{F} \end{pmatrix}$$

such that $V_1V_2 = V_2V_1$. If the T_i's are bi-contractive or power bounded and bi-contractive then the V_i's can be chosen to be bi-contractive or power bounded and bi-contractive also.

Here and below $*$ in a matrix representation stands for an operator which does not play a role in the discussion and does not need to be specified any further.

Proof of Lemma 2.1. Denote the inner product on the Krein space $\mathscr{K}$, by $[\cdot,\cdot]$ and fix a fundamental symmetry J on $\mathscr{K}$. Let $\{\mathscr{H}, (\cdot,\cdot)\}$ be the Hilbert space with $\mathscr{H} = \mathscr{K}$ as linear manifolds and $(x, y) = [Jx, y]$, $x, y \in \mathscr{K}$. Define the Hilbert space $\mathscr{F}$ by

$$\mathscr{F} = \mathscr{H}_1 \oplus \mathscr{H}_2 \oplus \cdots \oplus \mathscr{H}_n \oplus \cdots, \quad \mathscr{H}_n := \mathscr{H},\ n = 1, 2, \ldots$$

and define the Krein space $\widetilde{\mathscr{K}}$ as in the lemma. For each $i = 1, 2$, let D_{T_i} be the defect operator of T_i on $\mathscr{H}$:

$$D_{T_i} := (J - T_i^* J T_i)^{1/2}.$$

Then the operator W_i defined by

$$W_i(h_0, h_1, h_2, \ldots) = (T_i h_0, D_{T_i} h_0, 0, h_1, h_2, \ldots), \quad (h_0, h_1, h_2, \ldots) \in \widetilde{\mathscr{K}},$$

is an isometry on $\widetilde{\mathscr{K}}$. We identify $\widetilde{\mathscr{K}}$ with the space $\mathscr{K} \oplus \mathscr{E} \oplus \mathscr{E} \oplus \cdots$ where $\mathscr{E}$ is the Hilbert space

$$\mathscr{E} = \mathscr{H} \oplus \mathscr{H} \oplus \mathscr{H} \oplus \mathscr{H}$$

and define on $\widetilde{\mathscr{K}}$ a unitary operator $\widetilde{\mathscr{U}}$ of the form

$$\widetilde{U}(h_0, (h_1, h_2, h_3, h_4), \ldots) = (h_0, U(h_1, h_2, h_3, h_4), \ldots),$$

where U is a unitary operator on $\mathscr{E}$ which we define in two steps as follows.

(i) First we define U on the linear manifold

$$\mathscr{L} = \{(D_{T_1} T_2 h_0, 0, D_{T_2} h_0, 0) : h_0 \in \mathscr{H}\}$$

by the formula

$$U(D_{T_1} T_2 h_0, 0, D_{T_2} h_0, 0) = (D_{T_2} T_1 h_0, 0, D_{T_1} h_0, 0).$$

From the equalities

$$\begin{aligned}
&||(D_{T_1}T_2h_0, 0, D_{T_2}h_0, 0)||^2_{\mathscr{E}}\\
&\quad = ((J - T_1^*JT_1)^{1/2}T_2h_0, (J - T_1^*JT_1)^{1/2}T_2h_0)\\
&\qquad + ((J - T_2^*JT_2)^{1/2}h_0, (J - T_2^*JT_2)^{1/2}h_0)\\
&\quad = [h_0, h_0] - [T_2T_1h_0, T_2T_1h_0] = [h_0, h_0] - [T_1T_2h_0, T_1T_2h_0]\\
&\quad = ||(D_{T_2}T_1h_0, 0, D_{T_1}h_0, 0)||^2_{\mathscr{E}}
\end{aligned}$$

we see that U is a well-defined isometric operator on $\mathscr{L}$. We extend U by continuity to the closure $\bar{\mathscr{L}}$ of $\mathscr{L}$; we denote this extension also by U. Then $\overline{U\mathscr{L}} = \overline{U\mathscr{L}}$.

(ii) We extend U to a unitary operator from $\mathscr{E}$ onto itself by defining its action from $\mathscr{L}^\perp$ onto $(U\mathscr{L})^\perp$. To that end we first show that these spaces have the same dimension. If $\mathscr{H}$ is finite dimensional, then so is $\mathscr{E}$ and since U is an isometry, $\dim \mathscr{L}^\perp = \dim(U\mathscr{L})^\perp$. If $\mathscr{H}$ is infinite dimensional, then from the inequalities

$$\dim \mathscr{H} = \dim \varepsilon \geqslant \dim \mathscr{L}^\perp \geqslant \dim \mathscr{H}$$

we obtain $\dim \mathscr{L}^\perp = \dim \mathscr{H}$ and in the same way it can be proved that $\dim (U\mathscr{L})^\perp = \dim \mathscr{H}$. So, whatever the dimension of $\mathscr{H}$, we have

$$\dim \mathscr{L}^\perp = \dim(U\mathscr{L})^\perp. \tag{2.1}$$

Now choose an orthonormal basis $\{e_i\}$ of the subspace $\mathscr{L}^\perp$ and an orthonormal basis $\{f_i\}$ of the subspace $(U\mathscr{L})^\perp$ and extend U by linearity and continuity to an isometric operator on all of $\mathscr{E}$ by setting $Uei = f_i$. Equality (2.1) implies that U is unitary.

Having defined the unitary operator $\widetilde{\mathscr{U}}$ on $\widetilde{\mathscr{K}}$ we set $V_1 := \widetilde{U}W_1$ and $V_2 := W_2\widetilde{U}^{-1}$. Then V_1 and V_2 are isometric dilations of T_1 and T_2, respectively, and

$$\begin{aligned}
V_1V_2(h_0, h_1, \dots) &= (T_1T_2h_0, G(D_{T_1}T_2h_0, 0, D_{T_2}h_0, 0), (h_1, h_2, h_3, h_4), \dots)\\
&= (T_2T_1h_0, D_{T_2}T_1h_0, 0, D_{T_1}h_0, 0, h_1, h_2, h_3, h_4, \dots)\\
&= V_2V_1(h_0, h_1, \dots),
\end{aligned}$$

that is, $V_1V_2 = V_2V_1$.

The statement about the bi-contractive property follows from the fact that a contraction on a Krein space is a bi-contraction if and only if it maps a maximal uniformly negative subspace onto a maximal uniformly negative subspace (see, for example, [DR, Theorem 1.3.6]). For a proof of the statement concerning power boundedness we refer to the proof of [ABDJ, Theorem 2.5]. □

We now come to the main theorem of this section. It will be used in the proof of Theorem 3.4. It was shown in [ABDJ, Theorem 2.10] that if T is a bi-contraction in a Krein space $\mathscr{K}$ then it is power bounded if and only if there exist a maximal uniformly positive subspace $\mathscr{L}_+$ and a maximal uniformly negative subspace $\mathscr{L}_-$ which are invariant under T and such that $T|_{\mathscr{L}_-}$ is power bounded on $\mathscr{L}_-$. Since T is a bi-contraction we have moreover that $T\mathscr{L}_- = \mathscr{L}_-$ and the restriction $T|_{\mathscr{L}_-}$ is a bijection onto $\mathscr{L}_-$ (see, for example, [DR, Theorem 1.3,6]). The following theorem shows that if T_1 and T_2 are power-bounded bi-contractions which commute with each other then these invariant subspaces can be chosen to be the same.

Theorem 2.2. *For each $i = 1, 2$, let T_i be a power-bounded bi-contraction on a Krein space $\mathscr{K}$ and assume $T_1T_2 = T_2T_1$. Then there exist a maximal uniformly positive subspace $\mathscr{L}_+$ and a maximal uniformly negative subspace $\mathscr{L}_-$ which are invariant under T_1 and T_2.*

Proof. Let V_1 and V_2: $\widetilde{\mathscr{K}} \mapsto \widetilde{\mathscr{K}}$ be power-bounded bi-contractive isometric dilations of T_1 and T_2, respectively, such that $V_1V_2 = V_2V_1$ and $\widetilde{\mathscr{K}} \ominus \mathscr{K}$ is a Hilbert space. By Lemma 2.1 such dilations exist. For each $i = 1, 2$, consider the linear spaces

$$\mathscr{R}(V_i) := \bigcap_{k=1}^{\infty} \operatorname{ran} V_i^k, \quad \mathscr{L}(V_i) := \mathscr{R}(V_i)^{\perp} = \overline{\bigcup_{k=0}^{\infty} \ker V_i^{*k}}.$$

As shown in the proof of [ABDJ, Theorem 2.10], $\mathscr{R}(V_i)$ is a Krein subspace, $\mathscr{L}(V_i)$ is a Hilbert subspace of $\widetilde{\mathscr{K}}$, and both spaces are invariant under V_i. In fact, besides $V_i\mathscr{L}(V_i) \subset \mathscr{L}(V_i)$ we have $V_i\mathscr{R}(V_i) = \mathscr{R}(V_i)$ and $V_i|_{\mathscr{R}(V_i)}$ is unitary. That the two dilations commute implies

$$V_j\mathscr{R}(V_i) \subset \mathscr{R}(V_i), \quad j \neq i. \tag{2.2}$$

We claim that $\widetilde{\mathscr{K}}$ contains a maximal uniformly negative subspace $\widetilde{\mathscr{L}}_-$ which is invariant under V_1 and V_2. To see this we consider two cases.

(i) Assume V_1 is a unitary operator. Then $V_1V_2^* = V_2^*V_1$ which together with (2.2) imply $V_1\mathscr{R}(V_2) = \mathscr{R}(V_2)$ and $V_1\mathscr{L}(V_2) \subset \mathscr{L}(V_2)$. It follows that $W_i := V_i|_{\mathscr{R}(V_2)}$, $i = 1, 2$, are two commuting unitary operators on the Krein space $\mathscr{R}(V_2)$. By Phillips' theorem (see [Ph] and also [AI, Section 2, Corollary 5.20]), $\mathscr{R}(V_2)$ and then also $\widetilde{\mathscr{K}}$ contain a maximal uniformly negative W_i-invariant subspace which we denote by $\widetilde{\mathscr{L}}_-$. Since V_i is a bi-contraction we have $V_i\widetilde{\mathscr{L}}_- = \widetilde{\mathscr{L}}_-$, $i = 1, 2$, which proves the claim.

(ii) Assume V_1 is not unitary. Then the claim follows from (i) applied to the unitary operator $V_1|_{\mathscr{R}(V_1)}$ and the isometry $V_2|_{\mathscr{R}(V_1)}$ on the Krein space $\mathscr{R}(V_1)$ and the fact that, since $\mathscr{L}(V_1)$ is a Hilbert space, any maximal uniformly negative subspace of $\mathscr{R}(V_1)$ is a maximal uniformly negative subspace of $\widetilde{\mathscr{K}}$. This completes the proof of the claim.

Denote by $\widetilde{P}$ the projection in $\widetilde{\mathscr{K}}$ onto $\mathscr{K}$. Since $\widetilde{\mathscr{K}} \ominus \mathscr{K}$ is a Hilbert space and by [ABDJ, Lemma 2.8],[1] $\mathscr{L}_- := \widetilde{\mathscr{P}}\widetilde{\mathscr{L}}_-$ is a maximal uniformly negative subspace of $\mathscr{K}$ which is invariant under T_1 and T_2. If we repeat these arguments starting with the pair of commuting power-bounded bi-contractions T_1^* and T_2^* we can construct a maximal uniformly negative subspace $\mathscr{N}_-$ of $\mathscr{K}$ which is invariant under T_1^* and T_2^*. Then the subspace $\mathscr{L}_+ := \mathscr{K} \ominus \mathscr{N}_-$ is a maximal uniformly positive subspace of $\mathscr{K}$ which is invariant under T_1 and T_2. $\square$

We now formulate and prove a Krein space version of Ando's theorem in [A1], see also [SNF, Theorem 1.6.4].

Proposition 2.3. *For each $i = 1, 2$, let T_i be a bi-contraction on a Krein space $\mathscr{K}$ and assume $T_1T_2 = T_2T_1$. Then there are a Hilbert space $\mathscr{H}$ and for each i a unitary dilation U_i of T_i on the Krein space $\widetilde{\mathscr{K}} := \mathscr{K} \oplus \mathscr{H}$:*

$$U_i = \begin{pmatrix} T_i & 0 \\ * & * \end{pmatrix} : \begin{pmatrix} \mathscr{K} \\ \mathscr{H} \end{pmatrix} \to \begin{pmatrix} \mathscr{K} \\ \mathscr{H} \end{pmatrix}$$

such that $U_1U_2 = U_2U_1$.

We follow the proof [SNF, Theorem 1.6.4] and base the proof of this proposition on Lemma 2.1 and the following lemma.

Lemma 2.4. *For each $i = 1, 2$, let V_i be a bi-contractive isometry on a Krein space $\mathscr{K}$ and assume $V_1V_2 = V_2V_1$. Then there exist a Hilbert space $\mathscr{G}$ and for each i a unitary extension U_i of V_i on the Krein space $\widehat{\mathscr{K}} := \mathscr{K} \oplus \mathscr{G}$:*

$$U_i = \begin{pmatrix} V_i & * \\ 0 & * \end{pmatrix} : \begin{pmatrix} \mathscr{K} \\ \mathscr{G} \end{pmatrix} \to \begin{pmatrix} \mathscr{K} \\ \mathscr{G} \end{pmatrix}$$

such that $U_1U_2 = U_2U_1$. If the V_i's are power bounded, then the U_i's can be chosen to be power bounded also.

Proof. Since V_1 is a bi-contraction, there is a unitary extension U_1 of V_1 on a Krein space $\widehat{\mathscr{K}} \supset \mathscr{K}$ such that $\widehat{\mathscr{K}} \ominus \mathscr{K}$ is a Hilbert space and U_1 is minimal, which means that

$$\overline{\text{span}}\{U_1^n k : k \in \mathscr{K}, n \in \mathbb{Z}\} = \widehat{\mathscr{K}}.$$

[1] We use this opportunity to correct a mistake in the formulation of [ABDJ, Lemma 2.8]: The conclusions are valid if it as assumed that $\mathscr{G} \ominus \mathscr{K}$ is a Hilbert space. Then $\mathscr{G}^+ \ominus \mathscr{K}^+$ in the proof coincides with $\mathscr{G} \ominus \mathscr{K}$.

We denote the inner product on $\mathscr{K}$ as well as on $\widehat{\mathscr{K}}$ by $[\cdot,\cdot]$. For $f, g, \in \mathscr{K}$ and $n, m \in \mathbb{Z}$ we have

$$[U_1^n V_2 f, U_1^m V_2 g] = [U_1^n f, U_1^m g]. \tag{2.3}$$

Indeed, for $n \geqslant m$ we have

$$\begin{aligned}[U_1^n V_2 f, U_1^m V_2 g] &= [U_1^{n-m} V_2 f, V_2 g] = [V_1^{n-m} V_2 f, V_2 g] \\ &= [V_2 V_1^{n-m} f, V_2 g] = [V_1^{n-m} f, g] = [U_1^{n-m} f, g] = [U_1^n f, U_1^m g],\end{aligned}$$

the case $n < m$ can be proved by taking adjoints. In the following we denote by $\sum_n'$ an arbitrary sum over finitely many $n \in \mathbb{Z}$. Let $\mathscr{M}$ be the linear manifold in $\widehat{\mathscr{K}}$ defined by

$$\mathscr{M} = \left\{ \sum_n{}' \; U_1^n k_n : k_n \in \mathscr{K} \right\}.$$

It follows from (2.3) that (we use graph notation now)

$$U_2' := \left\{ \left\{ \sum_n{}' \; U_1^n k_n, \; \sum_n{}' \; U_1^n V_2 k_n \right\} : k_n \in \mathscr{K} \right\}$$

is an isometric relation on $\mathscr{M}$. To see that U_2' is an operator we need to verify

$$\sum_n{}' \; U_1^n k_n = 0 \Rightarrow \sum_n{}' \; U_1^n V_2 k_n = 0.$$

Assume the equality on the left-hand side and let $N \in \mathbb{N}$ be so large that for all indices n in this finite sum for which $k \neq 0$, we have $n + N \geqslant 0$. Then

$$\begin{aligned}U_1^N \sum_n{}' \; U_1^n V_2 k_n &= \sum_n{}' \; U_1^{n+N} V_2 k_n = \sum_n{}' \; V_1^{n+N} V_2 k_n \\ &= \sum_n{}' \; V_2 V_1^{n+N} k_n = V_2 U_1^N \sum_n{}' \; U_1^n k_n = 0\end{aligned}$$

and hence $\sum_n' U_1^n V_2 k_n = 0$. Since U_1 is minimal, we have that $\mathscr{M}$ is dense in $\widehat{\mathscr{K}}$. Hence U_2' is a densely defined isometric operator. Moreover, as V_2 is a bi-contraction and the restriction of U_2' to $\mathscr{K}$ coincides with $V_2 : U_2'|_{\mathscr{K}} = V_2$, we see that dom U_2' contains maximal uniformly negative subspaces of $\widehat{\mathscr{K}}$ (namely those of $\mathscr{K}$) whose image under U_2' are also maximal uniformly negative subspaces of $\widehat{\mathscr{K}}$. It follows (see [I], and also [AI, Theorem 2.4.6]) that U_2' can be extended by continuity to a bounded bi-contractive isometric operator on all of $\widehat{\mathscr{K}}$; we denote this extension by U_2' also. For all $k \in \mathscr{K}$ and $n \in \mathbb{Z}$ we have

$$U_1 U_2'(U_1^n k) = U_1 U_1^n V_2 k = U_1^{n+1} V_2 k = U_2' U_1^{n+1} k = U_2' U_1 (U_1^n k),$$

which, because of the minimality of U_1, implies that $U_1U_2' = U_2'U_1$. Now we consider two cases:

(i) V_2 is a unitary operator on $\mathscr{K}$. Then U_2' is a unitary operator on $\widehat{\mathscr{K}}$ and the proof of the lemma can easily be completed: Take $\mathscr{G} = \widehat{\mathscr{K}} \ominus \mathscr{K}$ and $U_2 = U_2'$.

(ii) V_2 is not unitary. Then we repeat the construction starting with U_2' and the unitary U_1 in place of V_1 and V_2, respectively. Then case (i) can be applied.

For the last statement we refer to the proof of [ABDJ, Theorem 2.5]. $\square$

Proof of Proposition 2.3. Consider the bi-contractive isometric dilations V_i of T_i on $\widetilde{\mathscr{K}} = \mathscr{K} \oplus \mathscr{F}$ from Lemma 2.1 and their unitary extensions U_i on $\widehat{\mathscr{K}} = \widetilde{\mathscr{K}} \oplus \mathscr{G}$ from Lemma 2.4 (with $\mathscr{K}$ replaced by $\widetilde{\mathscr{K}}$):

$$U_i = \begin{pmatrix} T_i & 0 & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} : \begin{pmatrix} \mathscr{K} \\ \mathscr{F} \\ \mathscr{G} \end{pmatrix} \to \begin{pmatrix} \mathscr{K} \\ \mathscr{F} \\ \mathscr{G} \end{pmatrix}, \quad i = 1, 2,$$

where $\mathscr{F}$ and $\mathscr{G}$ are Hilbert spaces. The matrix representation shows that U_i is a unitary dilation of T_i on $\mathscr{K} \oplus \mathscr{H}$ with $\mathscr{H} = \mathscr{F} \oplus \mathscr{G}$. Lemmas 2.1 and 2.4 imply that U_1 and U_2 commute. $\square$

3. Similarity and C_0-semigroups

The similarity theorem we prove in this section generalizes [ABDJ, Theorem 2.10], which concerns only one power-bounded bi-contraction, to a uniformly bounded C_0-semigroup of bi-contractions in a Krein space.

First we recall some definitions related to a semigroup $\{U(t)\}_{t>0}$ of bounded operators $U(t)$ on a Hilbert space $\{\mathscr{H}, (\cdot,\cdot)\}$. The semigroup is called a C_0-semigroup if it is strongly continuous in $t \in (0, \infty)$ and the strong limit $U(0) := s - \lim_{t\to 0} U(t)$ exists and $U(0) = I$. If $\{U(t)\}_{t>0}$ is a C_0-semigroup on $\mathscr{H}$, then the operator A on $\mathscr{H}$ defined by

$$\begin{cases} \operatorname{dom} A = \left\{ x \in \mathscr{H} : \lim\limits_{t\to 0} \dfrac{U(t) - I}{t} x \text{ exists} \right\}, \\ Ax = \lim\limits_{t\to 0} \dfrac{U(t) - I}{t} x, \quad x \in \operatorname{dom} A, \end{cases}$$

is a closed densely defined operator. It is called the generator of the C_0-semigroup. If $\{U(t)\}_{t>0}$ is a C_0-semigroup then there exist real numbers $M > 0$ and ω such that

$$\sup_{t>0} ||U(t)|| \leqslant Me^{\omega t} \tag{3.1}$$

and the generator A has the property

$$||(A-\lambda I)^{-n}||\leqslant\frac{M}{(\operatorname{Re}\lambda-\omega)^{n}},\quad \operatorname{Re}\lambda>\omega,\quad n=1,2,\dots\ . \tag{3.2}$$

Conversely, if A is a densely defined operator with this property, then it is the generator of a C_0-semigroup such that (3.1) holds. If $\omega\leqslant 0$ then the semigroup is called uniformly bounded. We recall the following result.

Lemma 3.1. *A C_0-semigroup $\{U(t)\}_{t>0}$ on a Hilbert space is uniformly bounded if and only if for some (and then for all) $t>0$, $U(t)$ is power bounded.*

Proof. If the semigroup is uniformly bounded, then for any $t>0$, $U(t)^n=U(nt)$, $n\in\mathbb{N}$, and from (3.1) with $\omega\leqslant 0$ it follows that $U(t)$ is power bounded. Assume that for $t_0>0$ the operator $U(t_0)$ is power bounded: $K:=\sup_{n\in\mathbb{N}}||U(t_0)^n||<\infty$. As the semigroup is C_0, $M:=\max_{0\leqslant s\leqslant t_0}||U(s)||<\infty$. For each $t>0$ there is an $n\in\mathbb{N}$ such that $nt_0\leqslant t<(n+1)t_0$, that is, $t=s+nt_0$ with $s=t-nt_0\in[0,t_0)$, and hence $||U(t)||=||U(s)U(t_0)^n||\leqslant MK$. $\square$

If (3.1) and (3.2) holds then $T=(A+(\omega+1)I)(A-(\omega+1)I)^{-1}$ is a bounded operator called the co-generator of the C_0-semigroup.

In a Krein space $\{\mathscr{K},[\cdot,\cdot]\}$ these notions also make sense provided we fix a fundamental symmetry J on the space and consider the corresponding Hilbert space inner product $(\cdot,\cdot)=[J\cdot,\cdot]$ and norm $||\cdot||$. This we shall do without saying this explicitly each time, as in the following proposition whose proof is similar to the proof of its Hilbert space version.

Proposition 3.2. *Let $\{U(t)\}_{t>0}$ be a C_0-semigroup of operators on a Krein space with generator A and co-generator T. Then the following statements are equivalent*:

(a) *For all $t>0$, $U(t)$ is a bi-contraction.*
(b) *A is maximal dissipative.*
(c) *T is a bi-contraction.*

Proof. (a) $\Rightarrow$ (b): Consider the identity with $x\in\mathscr{K}$ and $t>0$:

$$\begin{aligned}&[U(t)x-x,U(t)x-x]+[U(t)x-x,x]+[x,U(t)x-x]\\&\quad=[U(t)x,U(t)x]-[x,x].\end{aligned} \tag{3.3}$$

If we divide the expression on the left-hand side by t and let $t\to\infty$, then for $x\in$ dom A we obtain the limit $2\operatorname{Re}[Ax,x]$. As the expression on the right-hand side is nonpositive we see that $\operatorname{Re}[Ax,x]\leqslant 0$. Hence A is dissipative. $U(t)^*$ is also a C_0-semigroup of bi-contractions and its generator is A^* (this can be proved in the same way as in the Hilbert space case, see, for example, [B, Section 4, Theorem 4.3.1]);

thus A^* also is a dissipative operator. It follows (see, for example, [AI, Remark 2.2.7 and Definitions 2.1.1 and 2.2.1]) that A is maximal dissipative.

(b) $\Rightarrow$ (a): Since

$$\frac{d}{dt}[U(t)x, U(t)x] = 2\,\mathrm{Re}\,[AU(t)x, U(t)x] \leqslant 0, \tag{3.4}$$

the function $[U(t)x, U(t)x]$ is nonincreasing on $t>0$ and hence

$$[U(t)x, U(t)x] \leqslant [U(0)x, U(0)x] = [x, x],$$

that is, $U(t)$ is a contraction for $t>0$. By a similar argument, $U(t)^*$ is also a contraction. Thus (a) holds.

(b) $\Leftrightarrow$ (c): This follows from [AI, Theorem 2.6.13]. $\square$

The following proposition will be used in the proof of Theorem 4.4 in the next section.

Proposition 3.3. *Let $\{U(t)\}_{t>0}$ be a C_0-semigroup of bi-contractions in a Krein space.*

(a) *If for some $t_0>0$, $U(t_0)$ is unitary, then $U(t)$ is unitary for all $t>0$ and the semigroup can be extended to the C_0-group $\{\widetilde{U}(t)\}_{t\in\mathbb{R}}$ of unitary operators given by*

$$\widetilde{U}(t) = \begin{cases} U(t) & \text{if } t \geqslant 0, \\ U(-t)^{-1} & \text{if } t<0. \end{cases}$$

(b) *If, in addition, $U(t_0)$ is power bounded then the semigroup and the group are uniformly bounded.*

Proof. (a): For $n\in\mathbb{N}$, set $t_n = 2^{-n}t_0$. We claim $U(t_n)$ is unitary for every $n\in\mathbb{N}$. It suffices to prove the claim only for $n=1$. As $U(t_1)$ is a contraction,

$$[x,x] = [U(t_0)x, U(t_0)x] = [U(t_1)^2x, U(t_1)^2x] \leqslant [U(t_1)x, U(t_1)x] \leqslant [x,x],$$

which shows that $[U(t_1)x, U(t_1)x] = [x,x]$ and by polarization $[U(t_1)x, U(t_1)y] = [x,y]$ for all $x,y\in\mathscr{K}$. Hence $U(t_1)^*U(t_1) = I$. The same arguments apply to the adjoint operators and so $U(t_1)U(t_1)^* = I$. Thus $U(t_1)$ is unitary, and this proves the claim. Since

$$\lim_{n\to\infty}\frac{U(t_n)x - x}{t_n} = Ax, \quad x\in \mathrm{dom}\ A,$$

the argument in the proof of Proposition 3.2 starting with Eq. (3.3) shows that for $x\in\mathrm{dom}\ A$, $\mathrm{Re}[Ax,x] = 0$. Thus in (3.4) equality prevails, which implies that for all $t>0$,

$$[U(t)x, U(t)x] = [U(0)x, U(0)x] = [x,x],$$

first for all $x \in$ dom A and then by continuity for all $x \in \mathscr{K}$. Hence $U(t)^* U(t) = I$. The same arguments apply to $U(t)^*$, and so also $U(t)U(t)^* = I$. It follows that $U(t)$ is unitary. It is now easy to check that each $\widetilde{U}(t)$, $t \in \mathbb{R}$, is unitary and that they form a group. The strong continuity of this group follows from [HPh, Theorem 16.3.6].

(b): This follows from Lemma 3.1 and from

$$||\widetilde{U}(t)|| = ||U(-t)^*|| = ||U(-t)|| \leqslant \sup_{s>0} ||U(s)||, \quad t<0. \qquad \square$$

Theorem 3.4. *For a C_0-semigroup $\{U(t)\}_{t>0}$ of bi-contractions in a Krein space $\mathscr{K}$ the following statements are equivalent*:

(1) *$\{U(t)\}_{t>0}$ is uniformly bounded.*

(2) *There exist a maximal uniformly positive subspace $\mathscr{L}_+$ and a maximal uniformly negative subspace $\mathscr{L}_-$ of $\mathscr{K}$ which are invariant under $U(t)$ for all $t>0$ and the semigroup $\{U(t)|_{\mathscr{L}_-}\}_{t>0}$ is uniformly bounded on $\mathscr{L}_-$.*

(3) *$\{U(t)\}_{t>0}$ is similar to a C_0-semigroup of contractions in a Hilbert space.*

Proof. (1) $\Rightarrow$ (2): Fix an irrational number $t_0 \in (0,1)$ and set $T_1 = U(1)$ and $T_2 = U(t_0)$. By Theorem 2.2, there are a maximal uniformly positive subspace $\mathscr{L}_+$ and a maximal uniformly negative subspace $\mathscr{L}_-$ of $\mathscr{K}$ such that, $T_i\mathscr{L}_+ \subset \mathscr{L}_+$ and $T_i\mathscr{L}_- = \mathscr{L}_-$, $i = 1,2$. We show that these inclusions and equalities also hold with T_i replaced by $U(t)$, $t>0$. We denote by $[t]$ the integer part of $t(t-1<[t]\leqslant t)$ and by $\{t\} = t - [t]$ the fractional part of $t(\{t\} \in [0,1))$. Then for all $n \in \mathbb{N}$, we have $U(nt_0) = T_2^n$ and $U([nt_0]) = T_1^{[nt_0]}$, and hence

$$U(\{nt_0\})\mathscr{L}_- = U(\{nt_0\})U([nt_0])\mathscr{L}_- = U(nt_0)\mathscr{L}_- = \mathscr{L}_-.$$

We claim that $U(\{nt_0\})\mathscr{L}_+ \subset \mathscr{L}_+$. To see this, consider $x_+ \in \mathscr{L}_+$ and write $U(\{nt_0\})x_+ = y_+ + y_-$ with $y_\pm \in \mathscr{L}_\pm$. Then

$$U(nt_0)x_+ = U([nt_0])U(\{nt_0\})x_+ = U([nt_0])y_+ + U([nt_0])y_-.$$

Hence

$$\mathscr{L}_+ \ni U(nt_0)x_+ - U([nt_0])y_+ = U([nt_0])y_- \in \mathscr{L}^-,$$

which shows that $U([nt_0])y_- = 0$. Since $U([nt_0])$ is a bi-contraction, it is a bijection on $\mathscr{L}_-$ and therefore $y_- = 0$, that is, $U(\{nt_0\})x_+ = y_+ \in \mathscr{L}_+$. Thus we have shown that $U(\{nt_0\})\mathscr{L}_\pm \subset \mathscr{L}_\pm$ for all $n \in \mathbb{N}$. By Kronecker's theorem (see, for example, [KN]), the set $\{\{nt_0\} : n \in \mathbb{R}\}$ is dense in [0,1]. So, since $U(t)$ is strongly continuous, $U(t)\mathscr{L}_\pm \subset \mathscr{L}_\pm$ for all $t \in [0,1]$ and hence for all $t \in \mathbb{R}$. The last part of the statement in (2) follows directly from (1).

(2) $\Rightarrow$ (3): With $\mathscr{L}_\pm$ as in (2) we have $\mathscr{K} = \mathscr{L}_+ \dot{+} \mathscr{L}_-$. This is a result of Krein and Shmul'yan; see [AI, Corollary 1.8.6; A2, Corollary 1.5.2]. For $t \geqslant 0$, define $U_\pm(t) := U(t)|_{\mathscr{L}_\pm}$. Then the semigroup $\{U_+(t)\}_{t>0}$ consists of contractions in the Hilbert

space $\{\mathscr{L}_+, [\cdot,\cdot]\}$. As $U(t)$ is a bi-contraction, $U_-(t)\mathscr{L}_- = \mathscr{L}_-$, $t>0$. For $t\leqslant 0$, define $U_-(t) = U_-(-t)^{-1}$.

Then $\{U_-(t)\}_{t\in\mathbb{R}}$ is a group on the Hilbert space $\{\mathscr{L}_-, -[\cdot,\cdot]\}$ which extends the semigroup $\{U_-(t)\}_{t>0}$ on this space. That the group is strongly continuous follows from [HPh, Theorem 16.3.6]. Moreover, this group is uniformly bounded because, by assumption, $\{U_-(t)\}_{t>0}$ is uniformly bounded, and because of the uniform bound

$$-[U_-(t)x, U_-(t)x] \leqslant -[x,x], \quad t\leqslant 0,\ x\in\mathscr{L}_-,$$

which holds since $U(-t)$ is a contraction on $\{\mathscr{L}_-, [\cdot,\cdot]\}$. By a generalization to groups of a theorem of Sz.-Nagy (see [DK, Chapter I, Theorems 6.2 and 6.2′] and also [AI, 2.5.18]), this group is similar to a group $\{V(t)\}_{t\in\mathbb{R}}$ of contractions acting in the Hilbert space $\{L_-, -[\cdot,\cdot]\}$. Hence the semigroup $\{U_-(t)\}_{t>0}$ is similar to the semigroup $\{V(t)\}_{t>0}$. It follows that the semigroup $\{U(t)\}_{t>0}$ is similar to semigroup $\{U_+(t)\oplus V(t)\}_{t>0}$ of contractions in the Hilbert space $\mathscr{L}_+\oplus\mathscr{L}_-$ equipped with the inner product

$$(x,y) = [x_+, y_+] - [x_-, y_-], \quad x = x_- + x_+,\ y = y_- + y_+,\ x_\pm, y_\pm \in \mathscr{L}_\pm.$$

(3) $\Rightarrow$ (1): This follows simply because a semigroup of contractions in a Hilbert space is uniformly bounded by 1. $\square$

Remark 3.5. (a) By Lemma 3.1 and [ABDJ, Theorem 2.10], the statements (1)–(3) of Theorem 3.4 are also equivalent to the following three statements:

(4) For some $t>0$, $U(t)$ is power bounded.

(5) For some $t>0$, there are a maximal uniformly positive subspace $\mathscr{L}_+$ and a maximal uniformly negative subspace $\mathscr{L}_-$ of $\mathscr{K}$ which are invariant under $U(t)$ and the semigroup $\{U(t)|\mathscr{L}_-\}_{t>0}$ is power bounded.

(6) For some $t>0$, $U(t)$ is similar to a contraction in a Hilbert space.

(b)The similarity operator S can be chosen such that

$$\max\{||S||\,||S^{-1}||\}\leqslant\sqrt{4c^2+3} \tag{3.5}$$

with $c = \sup_{t>0}||U(t)||$, where, as always in the sequel, the norms are computed with respect to a preassigned fundamental decomposition of the Krein space.

Corollary 3.6. *Assume A is a closed densely defined and maximal dissipative operator on a Krein space. Then A is similar to a maximal dissipative operator in a Hilbert space if and only if $\sigma(A)\subset\{\lambda\in\mathbb{C} : \operatorname{Re}\lambda\leqslant 0\}$ and for some $M>0$*

$$||(A-\lambda I)^{-n}||\leqslant\frac{M}{(\operatorname{Re}\lambda)^n}, \quad \operatorname{Re}\lambda>0,\ n=1,2,\ldots, \tag{3.6}$$

where the norm is computed relative to a fixed fundamental decomposition of $\mathscr{K}$.

Proof. Assume A in the Krein space $\mathscr{K}$ is similar to a maximal dissipative operator A_0 on a Hilbert space $\mathscr{H}$ with similarity operator $S : \mathscr{K} \to \mathscr{H}$. Then $\sigma(A_0) \subset \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \leqslant 0\}$ and for A_0 the estimate (3.6) with the norm of $\mathscr{H}$ and $M = 1$ holds (see, for example, [AI, Lemma 2.2.8]). It follows that $\sigma(A) \subset \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \leqslant 0\}$ and (3.6) holds for A with $M = ||S||\,||S^{-1}||$. As to the converse, the inclusion and the estimate (3.6) imply that A generates a uniformly bounded C_0-semigroup (see [HPh, Theorem 12.3.1]). By Proposition 3.2, this semigroup consists of bi-contractions, and so by Theorem 3.4, it is similar to a semigroup of contractions on a Hilbert space. From the definition of a generator, it follows that A is similar to the generator of this semigroup of Hilbert space contractions, which is maximal dissipative. $\square$

We can add to the list with the 6 equivalent statements from Theorem 3.4 and Remark 3.5 another one.

Corollary 3.7. *Let $\{U(t)\}_{t>0}$ be a C_0-semigroup of bi-contractions on a Krein space $\mathscr{K}$. The following statements are equivalent*:

(1) *$\{U(t)\}_{t>0}$ is uniformly bounded.*

(7) *There is a Hilbert space $\mathscr{H}$ and a uniformly bounded C_0-group $\{\widetilde{U}(t)\}_{t\in\mathbb{R}}$ of unitary operators on $\widetilde{\mathscr{K}} = \mathscr{K} \oplus \mathscr{H}$ such that for each $t>0$ $\widetilde{U}(t)$ is a unitary dilation of $U(t)$.*

Proof. Assume (1). Then the generator A of the semigroup is a (closed densely defined and) maximal dissipative operator on $\mathscr{K}$ and satisfies (3.6). By Corollary 3.6, the generator A is similar to a maximal dissipative operator A_0 in a Hilbert space, hence their Cayley transforms $T = (A+I)(A-I)^{-1}$ and $T_0 = (A_0+I)(A_0-I)^{-1}$ are also similar. Since T_0 is a contraction, the bi-contraction T is power bounded. Hence T has a power-bounded unitary dilation $\tilde{T}$ on a Krein space $\widetilde{\mathscr{K}}$ such that $\mathscr{H} = \widetilde{\mathscr{K}} \ominus \mathscr{K}$ is a Hilbert space, which is similar to a unitary operator in a Hilbert space. If $\widetilde{A} := (\tilde{T}+I)(\tilde{T}-I)^{-1}$, then $i\widetilde{A}$ is a self-adjoint operator $\widetilde{\mathscr{K}}$, similar to a self-adjoint operator in a Hilbert space, and consequently, $\widetilde{A}$ is the generator of a uniformly bounded C_0-group $\{\widetilde{U}(t)\}_{t\in\mathbb{R}}$ of unitary operators on $\widetilde{\mathscr{K}}$. For each $t>0$, $\widetilde{U}(t)$ is a unitary dilation of $U(t)$.

If (7) holds then (1) follows from the equality $U(t) = P\widetilde{U}(t)|_{\mathscr{K}}$, where P is the projection in $\widetilde{\mathscr{K}}$ onto $\mathscr{K}$. $\square$

4. The condition H

Following [AI] we shall say that a bounded operator T on a Krein space $\mathscr{K}$ belongs to the class **H** if

$(\mathbf{H})_1$ there exist a T-invariant maximal nonnegative subspace and a T-invariant maximal nonpositive subspace of $\mathscr{K}$, and

$(\mathbf{H})_2$ every T-invariant maximal semi-definite subspace $\mathscr{L}$ admits a direct sum decomposition of the form

$$\mathscr{L} = \mathscr{L}^0 \dot{+} \widehat{\mathscr{L}}, \tag{4.1}$$

where the isotropic subspace $\mathscr{L}^0 := \mathscr{L} \cap \mathscr{L}^{\perp}$ is finite dimensional and the subspace $\widehat{\mathscr{L}}$ is uniformly definite.

Note: A T-invariant maximal semi-definite subspace $\mathscr{L}$ of $\mathscr{K}$ may have many direct sum decompositions of form (4.1) in which $\mathscr{L}^0$ is the isotropic subspace and $\widehat{\mathscr{L}}$ is a closed linear manifold. In each such decomposition the subspace $\widehat{\mathscr{L}}$ is necessarily definite. If $(\mathbf{H})_2$ holds, that is, if in one such decomposition $\widehat{\mathscr{L}}$ is uniformly definite, then in all these decompositions $\widehat{\mathscr{L}}$ is uniformly definite.

In the sequel we shall use the following examples.

Proposition 4.1. (a) *A bi-contraction on a Pontryagin or anti-Pontryagin space belongs to the class* $\mathbf{H}$.

(b) *Let T be a bi-contraction on a Krein space $\mathscr{K}$ and let $\mathscr{K} = \mathscr{K}_1 \oplus \mathscr{K}_2$ be an orthogonal decomposition of $\mathscr{K}$ in T-invariant Krein subspaces $\mathscr{K}_j$, $j = 1, 2$. If T belongs to the class $\mathbf{H}$ on $\mathscr{K}$ and the restriction $T|_{\mathscr{K}_1}$ belongs to the class $\mathbf{H}$ on $\mathscr{K}_1$, then the restriction $T|_{\mathscr{K}_2}$ belongs to the class $\mathbf{H}$ on $\mathscr{K}_2$.*

(c) *Let T be a bi-contraction on a Krein space $\mathscr{K}$ and assume that*

$$\mathscr{K} = \mathscr{P}_1 \oplus \mathscr{P}_2 \oplus \cdots \oplus \mathscr{P}_k \oplus \widehat{\mathscr{K}}$$

is an orthogonal decomposition of $\mathscr{K}$ into T-invariant Pontryagin or anti-Pontryagin subspaces $\mathscr{P}_j$, $j = 1, 2, \ldots, k$, and a T-invariant Krein subspace $\widehat{\mathscr{K}}$. If T belongs to the class $\mathbf{H}$ on $\mathscr{K}$, then the restriction $T|_{\widehat{\mathscr{K}}}$ belongs to the class $\mathbf{H}$ on *$\widehat{\mathscr{K}}$.*

Proof. (a): See [AI, Example 3.5.22].

(b): The proof uses that, since T is a bi-contraction and from class $\mathbf{H}$, every T-invariant nonnegative subspace can be extended to a T-invariant maximal nonnegative subspace and every T-invariant nonpositive subspace $\mathscr{L}_-$ such that $T\mathscr{L}_- = \mathscr{L}_-$ can be extended to a T-invariant maximal nonpositive subspace; see [AI, Corollary 3.5.3]. Let $\mathscr{K}_j = \mathscr{K}_{j+} \oplus \mathscr{K}_{j-}$ be a fundamental decomposition of $\mathscr{K}_j$, $j = 1, 2$. Then $\mathscr{K} = \mathscr{K}_+ \oplus \mathscr{K}_-$ with $\mathscr{K}_\pm = \mathscr{K}_{1\pm} \oplus \mathscr{K}_{2\pm}$ is a fundamental decomposition of $\mathscr{K}$. The Hilbert space structures defined by these decompositions are compatible; below when we say that the angle operators are contractions, we consider the summands with their Hilbert space inner products. We write T_j for the restrictions $T|_{\mathscr{K}_j}$ and we denote by P_j the projections in $\mathscr{K}$. onto $\mathscr{K}_j$, $j = 1, 2$.

We show that $\mathscr{K}_2$ contains a T_2-invariant maximal nonnegative subspace. Let $\mathscr{L}_{1+}$ be a T_1-invariant (and hence T-invariant) maximal nonnegative subspace of $\mathscr{K}_1$ (which exists on account of T_1 having property $(\mathbf{H})_1$) and let S_1 be its angle operator, that is, $S_1 : \mathscr{K}_{1+} \to \mathscr{K}_{1-}$ is a contraction and

$$\mathscr{L}_{1+} = \left\{ \begin{pmatrix} x_+ \\ S_1 x_+ \end{pmatrix} \in \begin{pmatrix} \mathscr{K}_{1+} \\ \mathscr{K}_{1-} \end{pmatrix} : x_+ \in \mathscr{K}_{1+} \right\}.$$

Let $\mathscr{L}_+$ be a T-invariant maximal nonnegative subspace extension of $\mathscr{L}_{1+}$. Then its angle operator $S : \mathscr{K}_+ \to \mathscr{K}_-$ is an extension of S_1, and so

$$\mathscr{L}_+ = \mathscr{L}_{1+} \dot{+} \left\{ \begin{pmatrix} x_+ \\ S x_+ \end{pmatrix} \in \begin{pmatrix} \mathscr{K}_+ \\ \mathscr{K}_- \end{pmatrix} : x_+ \in \mathscr{K}_{2+} \right\}.$$

We claim $\mathscr{L}_{2+} := P_2 \mathscr{L}_+$ is a T_2-invariant maximal nonnegative subspace of $\mathscr{K}_2$. Indeed,

$$T_2 \mathscr{L}_{2+} = T_2 P_2 \mathscr{L}_+ = T P_2 \mathscr{L}_+ = P_2 T \mathscr{L}_+ \subset P_2 \mathscr{L}_+ = \mathscr{L}_{2+}$$

and $\mathscr{L}_{2+}$ is maximal nonnegative as its angle operator $P_2 S|_{\mathscr{K}_{2+}} : \mathscr{K}_{2+} \to \mathscr{K}_{2-}$ is an everywhere defined contraction. In the same way it can be shown that $\mathscr{K}_2$ contains a T_2-invariant maximal nonpositive subspace. Now one starts with a T_1-invariant maximal nonpositive subspace $\mathscr{L}_{1-}$ of $\mathscr{K}_1$. Since T_1 is a bi-contraction we have that $T\mathscr{L}_{1-} = T_1 \mathscr{L}_{1-} = \mathscr{L}_{1-}$, and so $\mathscr{L}_{1-}$ can be extended to T-invariant maximal nonpositive subspace of $\mathscr{K}$, etc. Thus T_2 has property $(\mathbf{H})_1$.

We now show that T_2 has property $(\mathbf{H})_2$. Assume $\mathscr{L}_{2+}$ is a T_2-invariant (and hence T-invariant) maximal nonnegative subspace of $\mathscr{K}_2$ and let

$$\mathscr{L}_{2+} = \mathscr{L}^0_{2+} \dot{+} \widehat{\mathscr{L}}_{2+}$$

be any direct sum decomposition in which $\mathscr{L}^0_{2+} = \mathscr{L}_{2+} \cap \mathscr{L}^{\perp}_{2+}$ is the isotropic part of $\mathscr{L}_{2+}$ and $\widehat{\mathscr{L}}_{2+}$ is a (necessarily positive) subspace of $\mathscr{K}_2$. Let $\mathscr{L}_+$ be a T-invariant maximal nonnegative subspace extension of $\mathscr{L}_{2+}$ in $\mathscr{K}$ and denote by $\mathscr{L}^0_+$ its isotropic part. Then, as T has property $(\mathbf{H})_2$, we have $\dim \mathscr{L}^0_+ < \infty$, and since $\mathscr{L}^0_{2+} \subset \mathscr{L}^0_+$, the subspace $\mathscr{L}^0_{2+}$ is finite dimensional also. Set

$$\widetilde{\mathscr{L}}_{2+} = \mathscr{L}^0_+ \dot{+} \widehat{\mathscr{L}}_{2+}$$

and let $\mathscr{M}$ be the orthogonal complement in the Hilbert space sense (relative to some fundamental decomposition of the Krein space) of $\widetilde{\mathscr{L}}_{2+}$ in $\mathscr{L}_+$. Then

$$\mathscr{L}_+ = \widetilde{\mathscr{L}}_{2+} \dot{+} \mathscr{M} = \mathscr{L}^0_+ \dot{+} (\widehat{\mathscr{L}}_{2+} \dot{+} \mathscr{M})$$

and $\widehat{\mathscr{L}}_{2+} \dot{+} \mathscr{M}$ is a closed linear manifold. By the note after the definition of property $(\mathbf{H})_2$, the subspace $\widehat{\mathscr{L}}_{2+} \dot{+} \mathscr{M}$ and hence the subspace $\widehat{\mathscr{L}}_{2+}$ is uniformly positive. The remaining case concerning the decomposition of a T_2-invariant maximal nonpositive subspace of $\mathscr{K}_2$ can be proved similarly.

(c): Follows from (a) and by applying (b) k times. $\square$

A bounded operator T on a Krein space $\mathscr{K}$ will be called decomposable if there exist a maximal uniformly positive subspace $\mathscr{L}_+$ and a maximal uniformly negative subspace $\mathscr{L}_-$ of $\mathscr{K}$ which are invariant under T. Note that in this case $\mathscr{K} = \mathscr{L}_+ \dot{+} \mathscr{L}_-$ (see [AI, Corollary 1.8.16]), and relative to this decomposition T has the matrix representation:

$$T = \begin{pmatrix} T_+ & 0 \\ 0 & T_- \end{pmatrix}, \quad T_\pm = T|_{\mathscr{L}_\pm}. \tag{4.2}$$

For example, by [ABDJ, Theorem 2.10], a power bounded bi-contraction on a Krein space is decomposable.

Proposition 4.2. *If $\{U(t)\}_{t>0}$ is a uniformly bounded C_0-semigroup of bi-contractions on a Krein space $\mathscr{K}$, then there exists a decomposable bi-contraction X on $\mathscr{K}$ belonging to the class* $\mathbf{H}$, *such that $U(t)X = XU(t)$ for all $t>0$.*

Proof. By Theorem 3.4(ii), $\mathscr{K}$ admits the direct sum decomposition $\mathscr{K} = \mathscr{L}_+ \dot{+} \mathscr{L}_-$ where $\mathscr{L}_\pm$ is a maximal uniformly positive/negative subspace invariant under $U(t)$ for all $t>0$. Set $\mathscr{K}_+ = \mathscr{L}_-^\perp$, then $\mathscr{K} = \mathscr{K}_+ \oplus \mathscr{L}_-$ is a fundamental decomposition of $\mathscr{K}$; we denote the corresponding Hilbert space inner product by $(\cdot,\cdot)$. Since $\mathscr{L}_+$ is maximal uniformly positive, it can be represented as

$$\mathscr{L}_+ = \left\{ \begin{pmatrix} x_+ \\ Sx_+ \end{pmatrix} \in \begin{pmatrix} \mathscr{K}_+ \\ \mathscr{L}_- \end{pmatrix} : x_+ \in \mathscr{K}_+ \right\},$$

where the angle operator $S : \{\mathscr{K}_+, (\cdot,\cdot)\} \to \{\mathscr{L}_-, (\cdot,\cdot)\}$ is a strict contraction: $\|S\| < 1$. We define the operator X on $\mathscr{K}$ by

$$X = \alpha \begin{pmatrix} 0 & 0 \\ S & -I \end{pmatrix} : \begin{pmatrix} \mathscr{K}_+ \\ \mathscr{L}_- \end{pmatrix} \to \begin{pmatrix} \mathscr{K}_+ \\ \mathscr{L}_- \end{pmatrix},$$

where α is a real number $\geqslant \|(I - SS^*)^{-1}\|^{1/2}$. We show that X has the properties mentioned in the proposition.

(i) X commutes with each $U(t)$: Since $U(t)\mathscr{L}_- \subset \mathscr{L}_-$ (in fact equality holds but we do not need this here), $U(t)$ has the representation

$$U(t) = \begin{pmatrix} U_{11}(t) & 0 \\ U_{12}(t) & U_{22}(t) \end{pmatrix} : \begin{pmatrix} \mathscr{K}_+ \\ \mathscr{L}_- \end{pmatrix} \to \begin{pmatrix} \mathscr{K}_+ \\ \mathscr{L}_- \end{pmatrix}.$$

From $U(t)\mathscr{L}_+ \subset \mathscr{L}_+$ it follows that

$$SU_{11}(t) = U_{12}(t) + U_{22}(t)S.$$

This equality readily implies that $U(t)X = XU(t)$.

(ii) X is decomposable: $X\mathscr{L}_+ = \{0\} \subset \mathscr{L}_+$ and $X\mathscr{L}_- = \mathscr{L}_-$.

(iii) X is a bi-contraction: We have

$$X^* = \alpha\begin{pmatrix} 0 & -S^* \\ 0 & -I \end{pmatrix}, \quad XX^* = \alpha^2\begin{pmatrix} 0 & 0 \\ 0 & I - SS^* \end{pmatrix}$$

and hence for all $x = x_+ + x_- \in \mathscr{K}$, $x_+ \in \mathscr{K}_+$, $x_- \in \mathscr{L}_-$,

$$\begin{aligned}[X^*x, X^*x] &= -\alpha^2((I - SS^*)x_-, x_-) = -\alpha^2||\sqrt{I - SS^*}x_-||^2 \\ &\leqslant -\frac{\alpha^2}{||(I - SS^*)^{-1/2}||^2}||x_-||^2 = -\frac{\alpha^2}{||(I - SS^*)^{-1}||}||x_-||^2 \\ &\leqslant -||x_-||^2 = [x_-, x_-] \leqslant [x, x].\end{aligned}$$

Hence X^* is a contraction. Since $X^*\mathscr{L}_- = \mathscr{L}_+^{\perp}$, which is a maximal uniformly negative subspace of $\mathscr{K}$, X^* and then also X is a bi-contraction.

(iv) X belongs to class $\mathbf{H}$: By (ii), X has property $(\mathbf{H}_1)$. We show that X has property $(\mathbf{H}_2)$ also. First, let $\mathscr{M}_+$ be an X-invariant maximal nonnegative subspace. Then $X\mathscr{M}_+ \subset \mathscr{M}_+ \cap \mathscr{L}_- = \{0\}$. Hence $\mathscr{M}_+ \subset \ker X = \mathscr{L}_+$. It follows that $\mathscr{M}_+ = \mathscr{L}_+$, so $\mathscr{M}_+$ is uniformly positive. Next, let $\mathscr{M}_-$ be an X-invariant maximal nonpositive subspace with angle operator K, that is, $K : \{\mathscr{L}_-, (\cdot,\cdot)\} \to \{\mathscr{K}_+, (\cdot,\cdot)\}$ is a contraction and

$$\mathscr{M}_- = \left\{\begin{pmatrix} Kx_- \\ x_- \end{pmatrix} \in \begin{pmatrix} \mathscr{K}_+ \\ \mathscr{L}_- \end{pmatrix} : x_- \in \mathscr{L}_-\right\}.$$

Then for all $x_- \in \mathscr{L}_-$,

$$X\begin{pmatrix} Kx_- \\ x_- \end{pmatrix} = \begin{pmatrix} 0 \\ (SK - I)x_- \end{pmatrix} \in \mathscr{M}_-,$$

which implies $K(SK - I) = 0$. Since $||SK|| \leqslant ||S|| < 1$, the operator $SK - I$ is invertible on $\mathscr{L}_-$. Hence $K = 0$, that is, $\mathscr{M}_- = \mathscr{L}_-$, so $\mathscr{M}_-$ is uniformly negative. $\square$

We analyze further the relation between a C_0-semigroup and decomposable, bi-contractions which belong to the class $\mathbf{H}$, and prepare for a kind of converse of the above result. We denote by $\sigma_0(T)$ the set of all eigenvalues μ of T for which the kernel $\ker\,(T - \mu I)$ contains a (nonzero) neutral element.

Lemma 4.3. (a) *If T is a decomposable bounded operator on a Krein space then for every $\mu \in \mathbb{C}$, the subspace $\ker\,(T - \mu I)$ is regular.*

(b) *If T is a decomposable bi-contraction on a Krein space then $\sigma_0(T) \subset \mathbb{T}$.*

(c) *If T is a decomposable bi-contraction and from class* $\mathbf{H}$ *on a Krein space $\mathscr{K}$, then*

(i) $\sigma_0(T) \subset \mathbb{T}$,

(ii) $\sigma_0(T)$ *is empty or at most a finite set*, $\sigma_0(T) = \{\mu_1, \mu_2, \dots, \mu_k\}$, *say*, *for some* $k \in \mathbb{N}$,

(iii) *for each* $j = 1, 2, \dots, k$, $\ker(T - \mu_j I)$ *is either a Pontryagin or an anti-Pontryagin subspace of* $\mathscr{K}$, *and*

(iv) $\mathscr{K}$ *can be decomposed as*

$$\mathscr{K} = \mathscr{P} \oplus (\mathscr{L}_+ \dotplus \mathscr{L}_-), \quad \mathscr{P} := \ker(T - \mu_1 I) \oplus \ker(T - \mu_2 I) \oplus \cdots \oplus \ker(T - \mu_k I), \tag{4.3}$$

where the subspaces $\mathscr{L}_+$ *and* $\mathscr{L}_-$ *are the unique* T*-invariant maximal uniformly positive and* T*-invariant maximal uniformly negative subspaces of* $\mathscr{K} \ominus \mathscr{P}$.

Proof. (a): It follows from decomposition (4.2) that

$$\ker(T - \mu I) = \ker(T_+ - \mu I) \dotplus \ker(T_- - \mu I).$$

The first summand is uniformly positive and the second is uniformly negative. From [IS] it follows that their sum is a regular subspace.

(b): In the notation of representation (4.2) of T and part (a), if $x = x_+ + x_-$ with $x_\pm \in \mathscr{L}_\pm$ is a nonzero neutral eigenvector corresponding to $\mu \in \sigma_0(T)$, then $x_\pm \neq 0$ and $T_\pm x_\pm = \mu x_\pm$. The operator T_+ is a contraction in a Hilbert space, hence $|\mu| \leqslant 1$. On other hand, T_- is an expansive operator on a Hilbert space, hence $|\mu| \geqslant 1$. This implies (b).

(c): To prove the four items we recall the following. Since T is a contraction, the inner product $[(I - T^*T)\cdot, \cdot]$ is nonnegative on $\mathscr{K}$, which means that the Cauchy inequality holds for this inner product. Hence for $x \in \ker(T - \mu I)$, $\mu \in \sigma_p(T) \cap \mathbb{T}$, and any $y \in \mathscr{K}$ we have

$$\begin{aligned} |[\mu T^* x - x, y]| &= |[(T^*T - I)x, y]| \\ &\leqslant [(T^*T - I)x, x][(T^*T - I)y, y] \\ &= (|\mu|^2 - 1)[x, x][(T^*T - I)y, y] = 0. \end{aligned}$$

Here we used that $|\mu| = 1$. It follows that $T^* x = \mu^* x$, that is, $\ker(T - \mu I) \subset \ker(T^* - \mu^* I)$. Since also T^* is a contraction, equality prevails, that is,

$$\ker(T - \mu I) = \ker(T^* - \mu^* I), \quad \mu \in \sigma_p(T) \cap \mathbb{T}. \tag{4.4}$$

Moreover,

$$\ker(T - \mu I) \perp \ker(T - \nu I), \quad \mu, \nu \in \sigma_p(T) \cap \mathbb{T}, \ \mu \neq \nu. \tag{4.5}$$

Indeed, if $Tx = \mu x$ and $Ty = vy$ with $\mu, v \in \sigma_p(T) \cap \mathbb{T}$ and $\mu \neq v$, then by (4.4)

$$\mu[x, y] = [Tx, y] = [x, T^* y] = [x, v^* y] = v[x, y]$$

and hence $[x, y] = 0$. We now prove the four items.

(i) This follows from (b).

(ii) Select from each eigenspace ker $(T - \mu I)$ for which $\mu \in \sigma_0(T)$ a neutral nonzero element x_μ and define

$$\mathcal{N} = \overline{\operatorname{span}}\, \{x_\mu : \mu \in \sigma_0(T)\}.$$

Then $T\mathcal{N} \subset \mathcal{N}$, and by (i) and (4.5), $\mathcal{N}$ is a neutral subspace of $\mathcal{K}$. Since T is from class **H** and on account of [AI, Corollary 3.5.5], we have that the subspace $\mathcal{N}$ is finite-dimensional. Hence $\sigma_0(T)$ is the empty or a finite set.

(iii) By (a) $\ker(T - \mu I)$ is a regular subspace. Again as T is from class **H** and by [AI, Corollary 3.5.5], any maximal neutral subspace of $\ker(T - \mu I)$ is finite dimensional. This implies (iii).

(iv) According to (4.4), $\mathcal{P}$ is invariant under T^*, which implies that $\widehat{\mathcal{K}} := \mathcal{K} \ominus \mathcal{P}$ is invariant under T. By Proposition 4.1(c), the restriction $\widehat{T} := T|_{\widehat{\mathcal{K}}}$ is a bi-contraction and belongs to the class **H** on $\widehat{\mathcal{K}}$. As also $\sigma_0(\widehat{T}) = \emptyset$, by [AI, Lemma 3.5.7], there is exactly one $\widehat{T}$-invariant maximal uniformly positive subspace $\mathcal{L}_+$ and exactly one $\widehat{T}$-invariant maximal uniformly negative subspace $\mathcal{L}_-$ of $\widehat{\mathcal{K}}$. This implies the unique decomposition of $\mathcal{K} \ominus \mathcal{P}$ described at the end of item (iv). $\square$

Theorem 4.4. *Let $\{U(t)\}_{t>0}$ be a uniformly bounded C_0-semigroup of bi-contractions on a Krein space $\mathcal{K}$.*

(a) *Let T be a decomposable bi-contraction on $\mathcal{K}$ belonging to the class* **H** *such that $U(t)T = TU(t)$ for all $t>0$. Then the subspaces in decomposition* (4.3) *described in Lemma* 4.3(c) *are invariant under $U(t)$ for all $t>0$.*

(b) *Assume, in particular, that for some t_0, $U(t_0)$ belongs to the class* **H**, *then* (a) *holds for $T = U(t_0)$ and moreover, each of the semigroups $\{U(t)|_{\mathcal{P}_j}\}_{t>0}$, $\mathcal{P}_j = \ker(U(t_0) - \mu_j I)$ can be extended to a uniformly bounded C_0-group of unitary operators on $\mathcal{P}_j$, $j = 1, 2, \dots, k$, and $\{U(t)|_{\mathcal{L}_-}\}_{t>0}$ can be extended to a uniformly bounded C_0-group on $\mathcal{L}_-$.*

Remark 4.5. (i) In case of part (b), it follows that the semigroup $\{U(t)\}_{t>0}$ is similar to a semigroup of contractions in a Hilbert space. This was already proved in Theorem 3.4, but for this special case we have obtained a proof which does not make use of the results from Section 2.

(ii) The existence of an operator T satisfying the hypotheses of part (a) follows from Proposition 4.2. Hence a decomposition of $\mathcal{K}$ into canonical parts as in Lemma 4.3(c) which are $U(t)$-invariant for all $t>0$ always exists.

(iii) Part (b) and Proposition 4.1(a) imply that every uniformly bounded C_0-semigroup $\{U(t)\}_{t>0}$ of bi-contractions on a Pontryagin or anti-Pontryagin space $\mathscr{K}$ give rise to a decomposition of $\mathscr{K}$ into canonical parts of the type described in (ii).

Proof of Theorem 4.4. (a): Since the restriction $T|_{\mathscr{P}_j}$ is the operator of multiplication by μ_j and $T^*|\mathscr{P}_j$ the operator of multiplication by μ_j^* and $U(t)$ commutes with T, the space $\mathscr{P}_j$ is invariant under $U(t)$ and $U(t)^*$, $j = 1, 2, \dots, k$ and all $t > 0$. It follows that $\mathscr{K} \ominus \mathscr{P}$ is also invariant under $U(t)$ (and $U(t)^*$) for all $t > 0$.

(b): Since $U(t_0)$ is a power bounded bi-contraction, it is decomposable. Hence (a) with $T = U(t_0)$ can be applied. The statement concerning $\{U(t)|_{\mathscr{P}_j}\}_{t>0}$ follows from Proposition 3.3 as $U(t_0)|_{\mathscr{P}_j}$ is the unitary operator of multiplication by μ_j. The proof of the statement about $\{U(t)|_{\mathscr{L}_-}\}_{t>0}$ is the same as in the proof of implication (2) $\Rightarrow$ (3) of Theorem 3.4. $\square$

5. Stability

A C_0-semigroup $\{U(t)\}_{t>0}$ of bounded operators on a Hilbert space $\mathscr{K}$ is called stable if for all $x \in \mathscr{K}$,

$$\lim_{t \to \infty} ||U(t)x|| = 0,$$

or equivalently, $\lim_{n \to \infty} ||U(n)x|| = 0$. The equivalence can be seen as follows: Evidently, the first limit implies the second. For the converse implication write $t = [t] + \{t\}$, where $[t]$ is the integer part of t and $\{t\} = t - [t] \in [0, 1)$. Then the second limit implies

$$||U(t)x|| = ||U(\{t\})U([t])x|| \leqslant \left(\sup_{\tau \in [0;1]} ||U(\tau)|| \right) ||U([t])x|| \to 0, \quad t \to \infty.$$

Theorem 5.1. *Let $\{U(t)\}_{t>0}$ a be a uniformly bounded C_0-semigroup of bi-contractions on a Krein space $\mathscr{K}$ and denote its generator by A. If $\sigma(A) \cap i\mathbb{R}$ is a countable set, then $\mathscr{K}$ admits the orthogonal decomposition $\mathscr{K} = \mathscr{K}_s \oplus \mathscr{K}_u$, in which*

(i) $\mathscr{K}_s$ is a $U(t)$-invariant Hilbert subspace on which $\{U(t)|_{\mathscr{K}_s}\}_{t>0}$ is a stable C_0-semigroup of contractions, and

(ii) $\mathscr{K}_u$ is a $U(t)$-invariant Krein subspace on which $\{U(t)|_{\mathscr{K}_u}\}_{t>0}$ is a uniformly bounded C_0-semigroup of unitary operators and can be extended to a uniformly bounded C_0-group of unitary operators.

This theorem, like Theorem 4.4(b), implies without recourse to results from Section 2 but with the help of the generalization to groups of a theorem of Sz.-Nagy as in [DK, Chapter I, Theorems 6.2 and 6.2′], that $\{U(t)\}_{t>0}$ is similar to a C_0-semigroup of contractions on a Hilbert space.

Proof of Theorem 5.1. The direct sum decomposition of $\mathscr{K}$ originates from [AB] where semigroups are studied in the setting of normed spaces. [AB, Corollary 2.6] applied to the present situation implies that

$$\mathscr{K} = \mathscr{K}_s \dotplus \mathscr{K}_u, \tag{5.1}$$

where

$$\mathscr{K}_s = \left\{ x \in \mathscr{K} : \lim_{t \to \infty} ||U(t)x|| = 0 \right\}, \quad \mathscr{K}_u = \overline{\operatorname{span}} \{\ker(A - \lambda I) : \lambda \in i\mathbb{R}\} \tag{5.2}$$

are closed subspaces of $\mathscr{K}$ which are invariant under $U(t)$ for all $t > 0$ and such that $\{U(t)|_{\mathscr{K}_u}\}_{t>0}$ can be extended to a group of bounded operators in $\mathscr{K}$. We show that

(a) $\mathscr{K}_s$ is a Hilbert subspace of $\mathscr{K}$,
(b) $\mathscr{K}_u = \mathscr{K}_s^{\perp}$ and hence a Krein subspace of $\mathscr{K}$, and
(c) the semigroup $\{U(t)|_{\mathscr{K}_u}\}_{t>0}$ consists of unitary operators and its group extension is a uniformly bounded C_0-group of unitary operators on $\mathscr{K}_u$.

(a): Since $U(1)$ is a power bounded bi-contraction, by [ABDJ, Theorem 2.10], $\mathscr{K}$ contains a $U(1)$-invariant maximal uniformly positive subspace $\mathscr{L}_+$ and $U(1)$-invariant maximal uniformly negative subspace $\mathscr{L}_-$, and hence $\mathscr{K} = \mathscr{L}_+ \dotplus \mathscr{L}_-$ and $U(n)\mathscr{L}_\pm \subset \mathscr{L}_\pm$. Define the subspace

$$\widetilde{\mathscr{K}_s} := \operatorname{span}\{\mathscr{K}_s, \mathscr{L}_+\}.$$

For every vector $x = x_s + x_+ \in \widetilde{\mathscr{K}_s}$ with $x_s \in \mathscr{K}_s$, and $x_+ \in \mathscr{L}_+$ and every $n \in \mathbb{N}$ we have

$$\begin{aligned}[x,x] &\geqslant [U(n)x, U(n)x] = [U(n)(x_s + x_+), U(n)(x_s + x_+)] \\ &= [U(n)x_s, U(n)x_s] + 2\operatorname{Re}[U(n)x_s, U(n)x_+] + [U(n)x_+, U(n)x_+] \\ &\geqslant [U(n)x_s, U(n)x_s] + 2\operatorname{Re}[U(n)x_s, U(n)x_+].\end{aligned}$$

Since x_s belongs to $\mathscr{K}_s$, both summands at the end tend to 0 as $n \to \infty$. Hence $\widetilde{\mathscr{K}_s}$, is a nonnegative linear manifold which contains the maximal uniformly positive subspace $\mathscr{L}_+$ and therefore $\widetilde{\mathscr{K}_s}$ is closed and the two subspaces coincide (see, for example, [AI, Excercise 1.5.4]). It follows that $\mathscr{K}_s \subseteq \mathscr{L}_+$ and $\mathscr{K}_s$ is a uniformly positive, that is, a Hilbert subspace.

(b): On account of Proposition 3.2, the generator A is maximal dissipative, and therefore by [AI, Corollary 2.2.17],

$$\ker(A - \lambda I) = \ker(A^* - \lambda^* I), \quad \lambda \in i\mathbb{R}. \tag{5.3}$$

Fix $t>0$. From [HPh, Theorem 16.7.2] we have for $\mu\in\mathbb{C}$,

$$\ker(U(t)-\mu I)=\overline{\text{span}}\,\{\ker(A-\lambda I):\lambda\in\mathbb{C},\quad \mu=e^{\lambda t}\}. \tag{5.4}$$

If μ on the left-hand side varies through all of $\mathbb{T}$, then λ in the set on the right-hand side varies over all of $i\mathbb{R}$ and hence, on account of the formula for $\mathscr{K}_u$ in (5.2),

$$\mathscr{K}_u=\overline{\text{span}}\,\{\ker(U(t)-e^{\lambda t}I):\lambda\in i\mathbb{R}\}. \tag{5.5}$$

The right-hand side is independent of $t>0$. For $\mu\in\mathbb{T}$ we also have

$$\begin{aligned}\ker(U(t)-\mu I)&=\overline{\text{span}}\,\{\ker(A^*-\lambda^* I):\lambda^*\in i\mathbb{R},\ \mu^*=e^{\lambda^* t}\}\\ &=\ker(U(t)^*-\mu^* I).\end{aligned} \tag{5.6}$$

Here for the first equality we used (5.3) and (5.4). For the second equality we used (5.4) applied to $U(t)^*$ and A^*: A^* is the generator of the C_0-semigroup $\{U(t)^*\}_{t>0}$, which, as already mentioned in the proof of Proposition 3.2, can be proved in the same way as in the Hilbert space case. It follows that

$$\mathscr{K}_u=\text{span}\,\{\ker(U(t)^*-e^{-\lambda t}I):\ \lambda\in i\mathbb{R}\}. \tag{5.7}$$

Now consider $x_s\in\mathscr{K}_s$, $\lambda\in i\mathbb{R}$, and $x_u\in\ker(U(t)^*-e^{-\lambda t}I)\subset\mathscr{K}_u$. Then

$$x_u=e^{n\lambda t}U(t)^{*n}x_u=e^{n\lambda t}U(nt)^*x_u$$

and

$$|[x_s,x_u]|=|[x_s,U(nt)^*x_u]|=|[U(nt)x_sx_u]|\to 0,\quad n\to\infty.$$

Hence $\mathscr{K}_u\subset\mathscr{K}_s^{\perp}$ and from (5.1) we infer that equality holds, that is, $\mathscr{K}_u=\mathscr{K}_s^{\perp}$.

(c): Let $\{U_u(t)\}_{t\in\mathbb{R}}$ be the group extension of $\{U(t)|_{\mathscr{K}_u}\}_{t>0}$; thus for $t>0$, $U_u(t)=U(t)|_{\mathscr{K}_u}$. The group properties and (5.6) imply that for $t>0$,

$$\ker(U_u(-t)-e^{-\lambda t})=\ker(U_u(t)^*-e^{-\lambda t}),\quad \lambda\in i\mathbb{R},$$

that is, for x in these kernels, $U_u(-t)x=U_u(t)^*x$. From (5.5) and (5.7) it follows that this equality holds for all $x\in\mathscr{K}_u$. Hence $U_u(t)^{-1}=U_u(-t)=U_u(t)^*$ for all $t>0$, but then also for $t<0$; so $U_u(t)$ is a unitary operator on $\mathscr{K}_u$ for all $t\in\mathbb{R}$. From $||U_u(-t)||=||U(t)^*||=||U(t)||$, $t>0$, it follows that the group is uniformly bounded. The strong continuity of the group follows from [HPh, Theorem 16.3.6]. $\square$

Remark 5.2. The theorem implies that A is similar to a maximal dissipative operator in a Hilbert space and that $\{U(t)\}_{t>0}$ is similar to the semigroup generated by this operator. Indeed, relative to the orthogonal decomposition $\mathscr{K}=\mathscr{K}_s\oplus\mathscr{K}_u$ we can write every $U(t)$ and A in the matrix form

$$U(t) = \begin{pmatrix} U_s(t) & 0 \\ 0 & U_u(t) \end{pmatrix}, \quad A = \begin{pmatrix} A_s & 0 \\ 0 & A_u \end{pmatrix}. \tag{5.8}$$

Here $\{U_s(t)\}_{t>0}$ is a semigroup of contractions on the Hilbert space $\{\mathscr{K}_s, [\cdot,\cdot]\}$. Hence (3.1) holds with $\omega = 0$: There is a constant $M_s > 0$ such that for all $\operatorname{Re}\lambda > 0$ and $n = 1, 2, \ldots,$

$$||(A_s - \lambda I)^{-n}|| \leqslant \frac{M_s}{(\operatorname{Re}\lambda)^n}.$$

By a generalization of Sz.-Nagy's theorem, there are a group $\{V(t)\}_{t\in\mathbb{R}}$ of unitary operators on a Hilbert space $\mathscr{H}$ and a similarity operator $S : \mathscr{K}_u \to \mathscr{H}$ such that

$$U_u(t) = S^{-1}V(t)S, \quad t \in \mathbb{R}.$$

If A_0 is the generator of $\{V(t)\}_{t\in\mathbb{R}}$ then iA_0 is a self-adjoint operator on $\mathscr{H}$ and it follows that for $\operatorname{Im}\lambda \neq 0$ and $n = 1, 2, \ldots,$

$$||(iA_0 - \lambda I)^{-n}|| \leqslant \frac{1}{|\operatorname{Im}\lambda|^n}.$$

From this and $A = S^{-1}A_0S$ it follows that for all $\operatorname{Re}\lambda > 0$ and $n = 1, 2, \ldots,$

$$||(A_u - \lambda I)^{-n}|| \leqslant \frac{M_u}{(\operatorname{Re}\lambda)^n}, \quad M_u = ||S||\,||S^{-1}||.$$

Combining these estimates of the resolvent of A_s and of the resolvent of A_u we obtain the same estimates but with constant $\max\{M_s, M_u\}$ for the resolvent of A. This implies that in the space $\mathscr{K}$ considered as a Hilbert space A is also the generator of a semigroup of contractions.

If the semigroup has a strong limit when $t \to \infty$ we get a particularly simple decomposition of the Krein space in terms of invariant subspaces.

Theorem 5.3. *Let $\{U(t)\}_{t>0}$ be a uniformly bounded semigroup of bi-contractions on the Krein space $\mathscr{K}$. Assume the strong limit $\lim_{t\to\infty} U(t)x$ exists for every $x \in \mathscr{K}$. Then $\mathscr{H}$ can be decomposed in the direct sum of a maximal uniformly positive subspace $\mathscr{L}_+$ and a maximal uniformly negative subspace $\mathscr{L}_-$ invariant under all $U(t)$ and $U(t)|_{\mathscr{L}_-} = I$.*

Proof. The operator B defined by the strong limit

$$Bx := \lim_{t\to\infty} U(t)x, \quad x \in \mathscr{K}, \tag{5.9}$$

is linear, everywhere defined, and bounded: $||B|| \leqslant \sup_{t\in\mathbb{R}} ||U(t)||$. Moreover, $BU(t) = B = U(t)B$ for all $t > 0$ because, for example,

$$BU(t)x = \lim_{s\to\infty} U(s)U(t)x = \lim_{s\to\infty} U(s+t)x = Bx, \quad x \in \mathscr{K}.$$

Hence B is projection, that is, $B^2 = B$. Since $U(1)$ is a power bounded bi-contraction, $\mathscr{K}$ contains a $U(1)$-invariant maximal uniformly positive subspace $\mathscr{L}_+$ and a $U(1)$-invariant maximal uniformly negative subspace $\mathscr{L}_-$. Then $\mathscr{K} = \mathscr{L}_+ \dotplus \mathscr{L}_-$ and $U(n)\mathscr{L}_\pm \subset \mathscr{L}_\pm$, $n \in \mathbb{N}$. The inclusions imply $B\mathscr{L}_\pm \subset \mathscr{L}_\pm$. We show

(a) $\ker B$ is a uniformly positive subspace,
(b) $U(t)\mathscr{L}_+ \subset \mathscr{L}_+$ for all $t>0$, and
(c) $U(t)|_{\mathscr{L}_-}$ for all $t>0$.

(a): Choose $x \in \ker B$ and write $x = x_+ + x_-$, $x_\pm \in \mathscr{L}_\pm$. Then $Bx_+ + Bx_- = Bx = 0$ implies $Bx_- = -Bx_+ \in \mathscr{L}_- \cap \mathscr{L}_+ = \{0\}$. From $Bx_- = 0$ it follows that

$$0 = [Bx_-, Bx_-] = \lim_{t \to \infty} [U(t)x_-, U(t)x_-] \leqslant [x_-, x_-] \leqslant 0.$$

Hence $x_- = 0$ and $\ker B \subset \mathscr{L}_+$ is a uniformly positive subspace.

(b): Choose $x \in \mathscr{L}_+$ and write $U(t)x = y_+ + y_-$ with $y_\pm \in \mathscr{L}_\pm$. Then $By_- = Bx - By_+ \in \mathscr{L}_+ \cap \mathscr{L}_- = \{0\}$. By (a), $y_- = 0$, and (b) follows.

(c): Choose $x \in \mathscr{L}_-$ and write $U(t)x = z_+ + z_-, z_\pm \in \mathscr{L}_\pm$. Then $B(x - z_-) = B(U(t)x - z_-) = Bz_+ \in \mathscr{L}_+ \cap \mathscr{L}_- = \{0\}$. Because of (a), $x - z_- = 0$ and hence $U(t)x - x = z_+ \in \mathscr{L}_+$. By letting $t \to \infty$ we obtain that $Bx - x \in \mathscr{L}_+ \cap \mathscr{L}_- = \{0\}$. Hence $Bx = x$ and so for all $t>0$,

$$U(t)x = U(t)Bx = Bx = x. \qquad \square$$

6. Power-bounded co-generator

We consider the following problem: Is the co-generator of a uniformly bounded C_0-semigroup of bounded operators on a Hilbert space power bounded? The answer is positive when the generator A is bounded or when the inverse A^{-1} also is a generator of a uniformly bounded C_0-semigroup on a Hilbert space. See Theorems 6.1 and 6.2. Our proofs make ample use of results of Gomilko [G1,G2]. Besides these results, the problem is still open.

Theorem 6.1. *If the generator A of a uniformly bounded C_0-semigroup on a Hilbert space is bounded, then the co-generator $T := (A+I)(A-I)^{-1}$ is power bounded.*

Proof. Assume that $\{U(t)\}_{t>0}$ is a uniformly bounded C_0-semigroup on a Hilbert space $\{\mathscr{H}, (\cdot,\cdot)\}$ whose generator A is a bounded operator. Gomilko in [G2] showed that the co-generator T satisfies the inequality

$$2\pi|(T^n x, y)| \leqslant \frac{r^{n+2}}{n+1} \int_{-\pi}^{\pi} \left| \left((T - re^{i\theta} I)^{-2} x, y \right) \right| |d\theta| < \infty, \quad x, y \in \mathscr{H},\ n \in \mathbb{N},\ r>1. \tag{6.1}$$

We claim that T is power bounded if

$$L_{x,y} := \sup_{r>1} (r^2-1)\int_{-\pi}^{\pi}\left|\left(\left(T - re^{i\theta}I\right)^{-2}x, y\right)\right||d\theta| < \infty, \quad x, y \in \mathscr{H}. \tag{6.2}$$

To see this we note that (6.1) and (6.2) imply that for all $x, y \in \mathscr{H}$ and $r > 1$,

$$2\pi|(T^n x, y)| \leqslant \frac{r^{n+2}}{(n+1)(r^2-1)} L_{x,y}.$$

From

$$\lim_{n\to\infty} \frac{r_n^{n+2}}{(n+1)(r_n^2-1)} = \frac{1}{2}e, \quad r_n = 1 + \frac{1}{n}, \tag{6.3}$$

it follows that for all $x, y \in \mathscr{H}$,

$$|(T^n x, y)| \leqslant \frac{e}{4\pi} L_{x,y}.$$

The claim now follows from the uniform boundedness principle.

To show that the numbers $L_{x,y}$ in (6.2) are finite if A is bounded, we use another result from Gomilko [G1]: Since A is the generator of a uniformly bounded (C_0)-semigroup, it holds that

$$M_{x,y} := \sup_{\alpha>0} \alpha \int_{\alpha-i\infty}^{\alpha+i\infty} |((A - zI)^{-2}x, y)|\,|dz| < \infty, \quad x, y \in \mathscr{K}. \tag{6.4}$$

Every complex number z on the line $\operatorname{Re} z = \alpha$ with $\alpha > 0$ can be represented in the form $z = \frac{1+\lambda}{1-\lambda}$, where λ lies on the circle $\Gamma_\alpha : |\lambda - \frac{\alpha}{1+\alpha}| = \frac{1}{1+\alpha}$. Via the substitution $z = \frac{1+\lambda}{1-\lambda}$ in the defining integral for $M_{x,y}$ (6.4) can be written as

$$M_{x,y} = \sup_{\alpha>0} \oint_{\Gamma_\alpha} \frac{2\alpha}{|1-\lambda|^2}\left|\left(\left(A - \frac{1+\lambda}{1-\lambda}I\right)^{-2} x, y\right)\right||d\lambda| < \infty, \quad x, y \in \mathscr{K}. \tag{6.5}$$

For arbitrary $x, y \in \mathscr{K}$ and $u = (A^* - I)^2 y$, $L_{x,y}$ can be written as

$$\begin{aligned} L_{x,y} &= \sup_{r>1} (r^2-1)\int_{-\pi}^{\pi} \left|\left(\left((A+I)(A-I)^{-1} - re^{i\theta}I\right)^{-2} x, y\right)\right|\ |d\theta| \\ &= \sup_{r>1} (r^2-1)\int_{-\pi}^{\pi} \left|\left((A-I)^2(A+I-re^{i\theta}(A-I))^{-2}x, y\right)\right|\ |d\theta| \\ &= \sup_{r>1} (r^2-1)\int_{-\pi}^{\pi} \left|\left(((1-re^{i\theta})A + (1+re^{i\theta})I)^{-2} x, u\right)\right||d\theta| \\ &= \sup_{r>1} \int_{-\pi}^{\pi} \frac{r^2-1}{|1-re^{i\theta}|^2}\left|\left(\left(A + \frac{1+re^{i\theta}}{1-re^{i\theta}}I\right)^{-2} x, u\right)\right|\ |d\theta| \end{aligned}$$

$$= \sup_{0<r<1} \int_{-\pi}^{\pi} \frac{1-r^2}{1-re^{-i\theta}} \left| \left(\left(A - \frac{1+re^{-i\theta}}{1-re^{-i\theta}} I \right)^{-2} x, u \right) \right| r^{-1} |dre^{-i\theta}|$$

$$= \sup_{0<r<1} \oint_{C_r} \frac{r^{-1}-r}{|1-\lambda|^2} \left| \left(\left(A - \frac{1+\lambda}{1-\lambda} I \right)^{-2} x, u \right) \right| \, |d\lambda|$$

$$= \sup_{0<r<1} \oint_{C_r} |g_{x,u}(\lambda)| \, |d\lambda|,$$

where

$$g_{x,u}(\lambda) := \frac{r^{-1}-r}{(1-\lambda)^2} \left(\left(A - \frac{1+\lambda}{1-\lambda} I \right)^{-2} x, u \right), \quad x, u \in \mathscr{H}$$

and C_r is the circle $|\lambda| = r$. We consider two cases.

(1) $0 < r_0 \leqslant r < 1$. For $r \in (0,1)$, we take $\alpha = \frac{1-r}{2(1+r)}$. Then the circle C_r lies inside the circle Γ_α and

$$\frac{1-r^2}{r} = \frac{1-r}{1+r} \cdot \frac{(1+r)^2}{r} \leqslant 2\alpha \frac{(1+r_0)^2}{r_0}.$$

The function $g_{x,u}(\lambda)$ is analytic on the interior I_α of the circle Γ_α and continuous on $I_\alpha \cup \Gamma_\alpha$; the continuity at $\lambda = 1$ follows from

$$\lim_{\lambda \to 1} g_{x,u}(\lambda) = \lim_{\lambda \to 1} (r^{-1} - r) \Big(((1-\lambda)A - (1+\lambda)I)^{-2} x, u \Big) = \frac{(r^{-1}-r)}{4} (x,u).$$

Applying a result from [H, Chapter 8] we obtain that

$$\oint_{C_r} |g_{x,u}(\lambda)| \, |d\lambda| \leqslant \frac{(1+r_0)^2}{r_0} \oint_{\Gamma_\alpha} \frac{4\alpha}{|1-\lambda|^2} \left| \left(\left(A - \frac{1+\lambda}{1-\lambda} I \right)^{-2} x, u \right) \right| \, |d\lambda|.$$

Hence with $\alpha_0 = \frac{1-r_0}{2(1+r_0)}$ and by (6.5),

$$\sup_{r_0 \leqslant r < 1} \oint_{C_r} |g_{x,u}(\lambda)| \, |d\lambda|$$
$$\leqslant 2 \frac{(1+r_0)^2}{r_0} \sup_{0<\alpha\leqslant\alpha_0} \oint_{\Gamma_\alpha} \frac{2\alpha}{|1-\lambda|^2} \left| \left(\left(A - \frac{1+\lambda}{1-\lambda} I \right)^{-2} x, y \right) \right| \, |d\lambda|$$
$$\leqslant 2 \frac{(1+r_0)^2}{r_0} M_{x,u} < \infty.$$

(2) $0<r\leqslant r_0$. From (3.2) and since for all $\lambda\in C_r$

$$\operatorname{Re}\frac{1+\lambda}{1-\lambda}=\frac{1-r^2}{1+r^2-2\operatorname{Re}\lambda}>0,$$

we have

$$\left\|\left(A-\frac{1+\lambda}{1-\lambda}I\right)^{-2}\right\|\leqslant M\,\frac{(1+r^2-2\operatorname{Re}\lambda)^2}{(1-r^2)^2}$$

Therefore,

$$\begin{aligned}
&\sup_{0<r\leqslant r_0}\oint_{C_r}|g_{x,u}(\lambda)|\ |d\lambda|\\
&\quad\leqslant\sup_{0<r\leqslant r_0}M\int_{-\pi}^{\pi}\frac{r^{-1}-r}{(1+r^2-2\operatorname{Re}\lambda)^2}\frac{(1+r^2-2\operatorname{Re}\lambda)^2}{(1-r^2)^2}\,||x||\ ||u||r\ |d\theta|\\
&\quad=M\sup_{0<r\leqslant r_0}\int_{-\pi}^{\pi}\frac{d\theta}{(1+r)(1-r)}\,||x||\ ||u||\leqslant\frac{2\pi M}{1-r_0}\,||x||\ ||u||<\infty.
\end{aligned}$$

Combining the last equalities in (1) and (2) we obtain that for all $x,y\in\mathscr{H}$ and $u=(A^*-1)^2y$,

$$\begin{aligned}
L_{x,y}&=\max\left\{\sup_{r_0\leqslant r<1}\oint_{C_r}|g_{x,u}(\lambda)|\ |d\lambda|,\quad\sup_{0<r\leqslant r_0}\oint_{C_r}|g_{x,u}(\lambda)|\ |d\lambda|\right\}\\
&\leqslant\max\left\{2\,\frac{(1+r_0)^2}{r_0}\,M_{x,u},\quad\frac{2\pi M}{1-r_0}||x||\ ||u||\right\}<\infty,
\end{aligned}$$

which we had set out to prove. □

The following result was obtained, independently of our work, by Gomilko who gave a different proof.

Theorem 6.2. *If A and its inverse are generators of uniformly bounded C_0-semigroups on a Hilbert space then $T:=(A+I)(A-I)^{-1}$ is power bounded.*

Proof. We use the same notation as in and continue the proof of the previous theorem. By the uniform bounded principle, the last inequality in that proof implies that for some constant $K>0$ and for all $x\in\mathscr{H}$ and all $y\in$ dom A^{*2} with $u=(A^*-I)^2y$

$$L_{x,y}\leqslant K||x||\ ||u||.$$

Combining this with (6.1) we obtain that

$$2\pi|((A-I)^{-2}T^n x, u)| \leqslant K \frac{r^{n+2}}{(n+1)(r^2-1)} ||x|| \; ||u||, \quad x, u \in \mathscr{H}, \; n \in \mathbb{N}, \; r > 1.$$

Since $(T-I)^2 = 4(A-I)^{-2}$ and by (6.3), there exists a number $\widetilde{K}$ such that for all $x, u \in \mathscr{H}$ and all $n \in \mathbb{N}$,

$$||((T-I)^2 T^n x, u)|| \leqslant \widetilde{K} \, ||x|| \; ||u||.$$

Let S be the co-generator of the semigroup of which A^{-1} is the generator. Then

$$S = (A^{-1}+I)(A^{-1}-I)^{-1} = (I+A)(I-A)^{-1} = -T.$$

Hence there is a constant $\widetilde{K}_1 > 0$ such that for all $x, u \in \mathscr{H}$ and all $n \in \mathbb{N}$,

$$||((T+I)^2 T^n x, u)|| \leqslant \widetilde{K}_1 ||x|| \; ||u||.$$

The power boundedness of T now follows from

$$4T^{n+1} = (T+I)^2 T^n - (T-I)^2 T^n. \quad \square$$

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